Laplace Equation

Consider the Dirichlet Problem $u_{xx} + u_{yy} = 0$, with u bounded in the half strip $\{0 \leq x \leq L\} \times \{y \geq 0\}$ and $u(0, y) = u(L, y) = 0$. Since the equation is homogeneous there will be an arbitrary constant in the solution. Assume the solution is of the form $u(x, y) = X(x)Y(y)$. Then

$$u_{xx}(x, y) = X''(x)Y(y) \tag{1}$$

and

$$u_{yy}(x, y) = X(x)Y''(y) \tag{2}$$

Solving (1) and (2) for a separated solution yields the linked homogeneous second order linear differential equations

$$X'' + \lambda X = Y'' - \lambda X = 0.$$

For $\lambda = 0$, the solution to these equations is the trivial solution $u \equiv 0$.

If u is not identically zero, substituting $x = 0, y = 0$ and $x = L, y = 0$ into (1)

$$X(0) = X(L) = 0. \tag{3}$$

For $\lambda > 0$, equation (1) has solutions

$$X(x) = Ae^{\sqrt{\lambda}x} + Be^{-\sqrt{\lambda}x},$$

and the only solution satisfying (3) is $X \equiv 0$ which gives the trivial solution $u \equiv 0$.

For $\lambda < 0$, equations (1) and (2) have solutions

$$X(x) = A\cos(\sqrt{|\lambda|}x) + B\sin(\sqrt{|\lambda|}x)$$

and

$$Y(y) = Ce^{\sqrt{|\lambda|}y} + De^{-\sqrt{|\lambda|}y}.$$

From (3) we conclude $A = 0$ and $L\sqrt{|\lambda|} = n\pi$ where n is an integer. Thus,

$$X(x) = B\sin\left(n\pi x/L\right).$$

Moreover, since Y is bounded on the positive half-line because u is bounded in the upper half-strip, we have $C = 0$ so that

$$Y(y) = De^{-n\pi y/L}.$$

Finally, we have

$$u(x, y) = \mathbf{C}\sin\left(n\pi x/L\right)e^{-n\pi y/L} \tag{4}$$

where n is an integer and $\mathbf{C} = BD$ is an arbitrary constant.

Every solution can be represented by (4) with appropriate choices of n and $\mathbf{C}$.