1 Second order linear constant coefficient differential equations

First consider the special case of second order linear differential equations with constant coefficients where the characteristic polynomial is quadratic and has two distinct roots or one root of multiplicity 2.

Theorem 1. *Suppose* a, b *and* c *are real numbers and* $a \neq 0$. *Then*

$$aX'' + bX' + cX = 0 \tag{1}$$

is a homogeneous second order linear differential equation with constant coefficients whose characteristic polynomial $ar^2 + br + c$ *has discriminant* $d = b^2 - 4ac$. *The solutions of (1) are given by*

$$X(x) = \begin{cases} e^{-bx/2a}(C_1 x + C_2) & \text{if } d = 0 \\ e^{-bx/2a}\left[C_1 e^{\sqrt{d}\,x/2a} + C_2 e^{-\sqrt{d}\,x/2a}\right] & \text{if } d \neq 0 \end{cases} \tag{2}$$

where C_1 *and* C_2 *are arbitrary constants.*

Corollary 1. *The solutions of (2) can also be written in terms of trigonometric and hyperbolic functions. There are two cases,* $d > 0$ *and* $d < 0$.

$$X(x) = \begin{cases} e^{-bx/2a}\left[C_1 \sinh(\sqrt{d}\,x/2a) + C_2 \cosh(\sqrt{d}\,x/2a)\right] & \text{if } d > 0 \\ e^{-bx/2a}\left[C_1 \sin(\sqrt{-d}\,x/2a) + C_2 \cos(\sqrt{-d}\,x/2a)\right] & \text{if } d < 0 \end{cases}$$

where C_1 *and* C_2 *are arbitrary constants.*

Example 1. *Consider the equation* $X'' + 2X' + \lambda X = 0$ *in which* $a = 1, b = 2, c = \lambda$ *and* $d = 4(1 - \lambda)$.

$$X(x) = \begin{cases} e^{-x}(C_1 x + C_2) & d = 0 \iff \lambda = 1 \\ e^{-x}\left(C_1 \sinh\sqrt{1-\lambda}\,x + C_2 \cosh\sqrt{1-\lambda}\,x\right) & d > 0 \iff \lambda < 1 \iff \sqrt{d}/2a = \sqrt{1-\lambda} \\ e^{-x}\left(C_1 \sin\sqrt{\lambda-1}\,x + C_2 \cos\sqrt{\lambda-1}\,x\right) & d < 0 \iff \lambda > 1 \iff \sqrt{-d}/2a = \sqrt{\lambda-1} \end{cases}$$

Example 2. *Consider the equation* $X'' + \lambda X = 0$ *in which* $a = 1, b = 0, c = \lambda$ *and* $d = -4\lambda$.

$$X(x) = \begin{cases} C_1 x + C_2 & d = 0 \iff \lambda = 0 \\ C_1 \sinh\sqrt{-\lambda}\,x + C_2 \cosh\sqrt{-\lambda}\,x & d > 0 \iff \lambda < 0 \iff \sqrt{d}/2a = \sqrt{-\lambda} \\ C_1 \sin\sqrt{\lambda}\,x + C_2 \cos\sqrt{\lambda}\,x & d < 0 \iff \lambda > 0 \iff \sqrt{-d}/2a = \sqrt{\lambda} \end{cases}$$

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2 Higher order constant coefficient linear differential equations

Now consider the more general case where the characteristic polynomial has degree n and there are n roots counted according to multiplicity.

Theorem 2. *Suppose $a_0, a_1, \ldots, a_n$ are real numbers and $a_n \neq 0$. then*

$$\sum_{j=0}^{n} a_j \frac{\partial^j X}{\partial^j x} = 0$$

is a homogeneous constant coefficient linear differential equation with characteristic polynomial

$$\sum_{j=0}^{n} a_j r^j \tag{3}$$

whose general solution is given by

$$X(x) = \sum_{j=1}^{n} C_j y_j(x)$$

where the C_j are arbitrary constants and to each distinct root, r of (3), the y_j are the functions defined by

- *letting, for each real root r of multiplicity k, the corresponding k functions $y_j = x^i e^{rx}$ for $i = 0, 1, 2, \ldots k-1$*
- *letting, for each pair of complex conjugate roots $r = a \pm bi$ of multiplicity k, the corresponding $2k$ functions $y_j = x^i e^{ax}\cos(bx)$ and $y_j = x^i e^{ax}\sin(bx)$ $i = 0, 1, 2, \ldots k-1$.*

Remark. *Occasionally it may be convenient to write the solution in terms of hyperbolic rather than exponential functions. This accomplished by using the identity $e^{\alpha} = \cosh\alpha + \sinh\alpha$.*

Example 3. *The characteristic polynomial of $X^{(6)} + 8X^{(4)} + 16X^{(2)} = 0$ is $r^2(r^2+4)$ with roots $0, 2i$ and $-2i$, each with multiplicity 2. The general solution is*

$$X(x) = C_1 + C_2 x + C_3\cos(2x) + C_4 x\cos(2x) + C_5\sin(2x) + C_6 x\sin(2x).$$

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